Module 1
Circles

What this module is about

This module will discuss in detail the characteristics of a circle as well as the segments and lines associated with it. Here, you will gain deeper understanding of the angles formed in circles, how to get their measures and how they are related to one another. Furthermore, this module will also give meaning to the circle being composed of arcs and how each arc is related to the angles formed in circles.

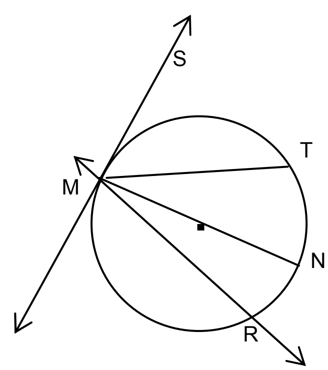
What you are expected to learn

This module is written for you to

1. define a circle.
2. define and show examples of the lines and segments associated with circles.
3. describe the relationship of lines and segments that are peculiar to circles.
4. define, identify and give examples of the kinds of arcs that compose a circle.
5. identify central angle and inscribed angle.
6. discover the relationship between the measures of central angle and inscribed angle and their intercepted arcs.

How much do you know

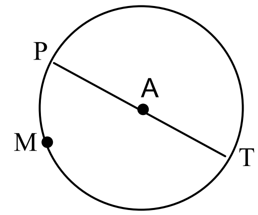
Answer the following as indicated.



1. Given a circle with center O. Name the following :
    a. the circle
    b. a diameter
    c. two radii
    d. two chords which are not diameters
    e. a secant
    f. a tangent

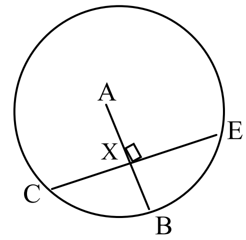
2. If a radius is perpendicular to a chord then it ________ the chord.

3. In the given circle A, $\overline{PT}$ is a diameter, therefore $\widehat{MT}$ is a ________ and
4. $\widehat{PTM}$ is a _________.

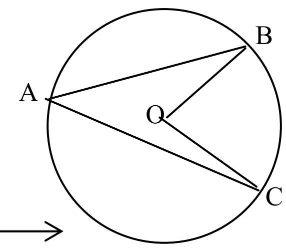


5. Radius AB $\perp$ CE. If $\overline{CE}$ = 8 cm, then $\overline{CX}$ = ________.

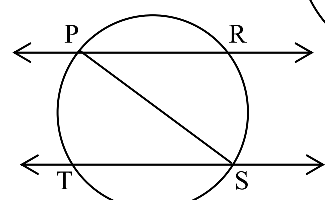
6. Using the same figure, if $\overline{AX}$ = 3 cm, What is the length of radius AC?



7. In circle O, $m\angle BOC = 93$. What is $m\widehat{BC}$?
8. What is $m\angle BAC$?



9. In the figure, $\overleftrightarrow{PR} \parallel \overleftrightarrow{ST}$. Using the given Find $m\widehat{PT}$ and $m\angle RPS$.



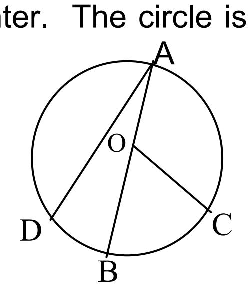
10. A quadrilateral PQRS is inscribed in a circle. If $m\angle P = 103$, what is $m\angle R$?

What you will do

Lesson 1

Identifying a circle, the lines, segments and angles associated with it.

A circle is defined as the set of all points that are at the same distance from a given point in the plane. The fixed given point is called the center. The circle is named after its center. Hence in the figure, given is a circle O.



The set of points on the plane containing the circle is divided into 3, (1) the circle, (2) the set of points outside the circle and (3) the set of points inside the circle. $\overline{OC}$, $\overline{OB}$ and $\overline{OA}$ are segment whose endpoints are the center of the circle and a point on the circle. These three segments are called radii of the circle.

Radius of a circle is a segment whose endpoints are the center and a point on the circle. In the figure, $\overline{AD}$ is a segment whose endpoints are points on the circle. $\overline{AD}$ is called chord of the circle. $\overline{AB}$ is a segment whose endpoints are points on the circle and it passes through the center. $\overline{AB}$ is called <u>diameter</u> of a circle. Diameter of a circle is a chord that passes through the center.

Lines on the plane containing the circle may intersect the circle at one point or at two points or not at all.

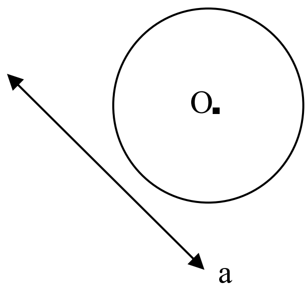


Fig. 1. line a does not intersect circle O.

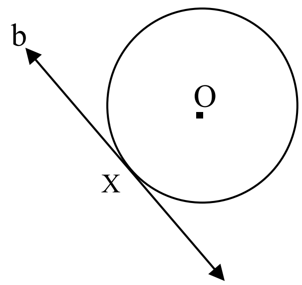


Fig. 2. line b intersect circle O at point X

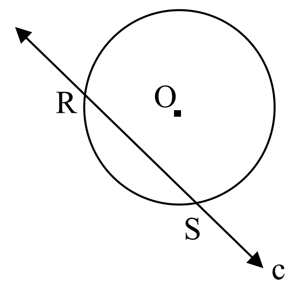


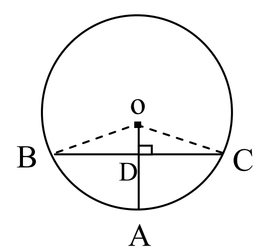
Fig. 3 line c intersect circle at two points R and S.

In figure 2, line b is tangent to the circle, and in figure 3, line c is a secant. Hence, we can use the following definitions.

Tangent is a line that intersect a circle at one points. Secant is a line that intersect a circle at two points.

Some theorems in circle show relationship between chord and radius. One of them is this theorem:

Theorem: If a radius is perpendicular to a chord, then it bisects the chord.



Proof: Consider the given circle. If radius $\overline{OA} \perp \overline{BC}$ at D, then $\overline{OA}$ bisects $\overline{BC}$ or BD = DC. One way of proving segments or angles congruent is by showing that they are corresponding parts of congruent triangles. Here, we must prove that $\overline{BD}$ and $\overline{DC}$ are corresponding sides of congruent triangles. If O and B are joined and O and C are also joined, we have ∆OBD and ∆OCD. Both of these triangles are right since $\overline{OA} \perp \overline{BC}$ and thus $\angle ODB$ and $\angle ODC$ are both right angles. Since $\overline{OB}$ and $\overline{OC}$ are both radii of the same circle, hence they are congruent. And finally $\overline{OD} \cong \overline{OB}$ by reflexive property. Therefore, by the HyL Congruency for right triangles, $\Delta OBD \cong \Delta OCD$. Since the two triangles are congruent, then the remaining corresponding parts such as $\overline{BD}$ and $\overline{DC}$ are also congruent.

We have just proven the theorem here, only this time, instead of using the two column form we use the paragraph form.

Our conclusion therefore is that a radius that is <u>perpendicular</u> to a chord <u>bisects</u> the chord. The most important considerations here were the perpendicularity and the word to bisect.

Examples:

1. $\overline{OB} \perp \overline{DE}$ at T, $\overline{DT}$ = 3x -7 , $\overline{TE}$ = x + 15

Solution:

Since $\overline{OB} \perp \overline{DE}$, then $\overline{DT} = \overline{TE}$

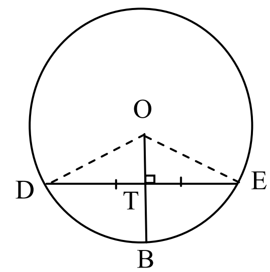
Hence,

3x – 7 = x + 15

2x = 15 + 7

2x = 22

x = 11



Substituting the value of x, we get

$\overline{DT}$ = 3(11) – 7 = 33 – 7

= 26

$\overline{TE}$ = 11 + 15 = 26

$\overline{DE} = \overline{DT} + \overline{TE}$

$\overline{DE}$ = 26 + 26 = 52

There are other theorems whose main idea is taken from the previously proven theorem. The next theorem serves as the converse of the first theorem and it states that: If a radius of a circle bisects a chord that is not a diameter, then it is perpendicular to the chord.

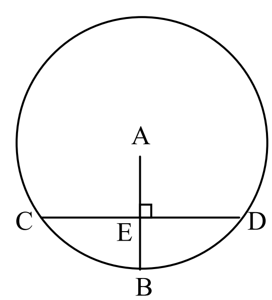
If the previous theorem was proven using the HyL congruence for right triangle, the converse is proven using the reverse process, that is two angles must be proven part of congruent triangles and they are congruent and supplementary.

You can prove the theorem as part of your exercise. Examples on how to use these two theorems are given below.

2. Given: $\overline{AB}$ bisects chord $\overline{CD}$ at E.

$\overline{CD}$ = 6, $\overline{AE}$ = 4

Find the length of the radius of the circle.



Solution: Based on the theorem, $\overline{AB} \perp \overline{CD}$, thus

$\Delta ACE \cong \Delta ADE$ and both are right triangles.

By the Pythagorean theorem, we can solve for the length of the radius.

In ΔACE, $\overline{AC}^2 = \overline{AE}^2 + \overline{CE}^2$

But $\overline{CE}$ = ½ $\overline{CD}$ so

$\overline{CE} = ½ (6)$
$\overline{CE} = 3$

$\overline{AC}^2 = \overline{AE}^2 + \overline{CE}^2$
$\overline{AC}^2 = 4^2 + 3^2$
$\overline{AC}^2 = 16 + 9 = 25$
$\overline{AC} = \sqrt{25}$
$\overline{AC} = 5$

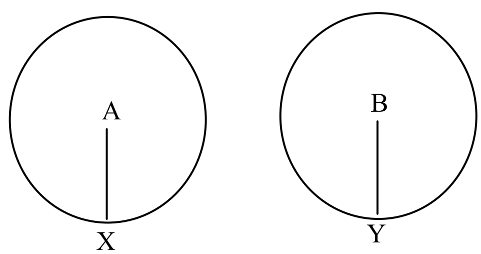
Lesson on circle is very rich with theorems and definitions, principles and postulates. Some of those theorems and definitions will be introduced as we plod along with this module.

Definitions:

- Congruent circles are circles that have congruent radii.
- Concentric circles are coplanar circles having the same center

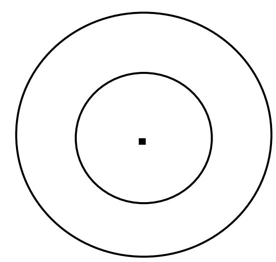
Illustrations:

a)



Circle A is congruent to circle B if and only if $\overline{AX} \cong \overline{BY}$

b)



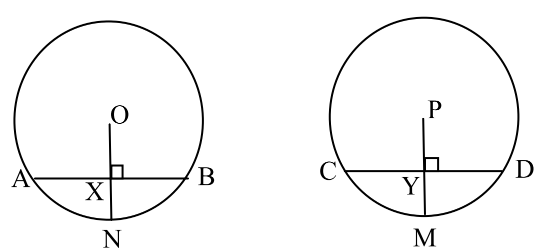
These two circles are concentric circles

Theorem:

If chords of a circle or of congruent circles are equidistant from the center(s), then the chords are congruent

Illustration of the theorem.

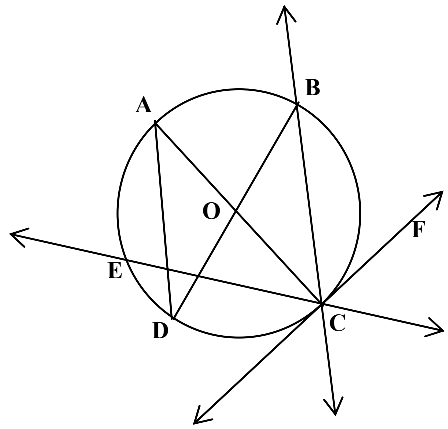
Circle O $\cong$ circle P
$\overline{OX} = \overline{PY}$
Then, $\overline{AB} \cong \overline{CD}$



Try this out

A. Using the given figure, name

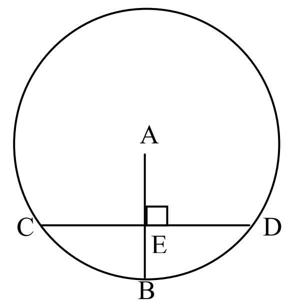
1. the circle
2. 2 diameters
3. 2 chords which are not diameters
4. 2 secants
5. a tangent



B. Given: $\overline{AB} \perp \overline{CD}$ at E

$\overline{CD}$ is 10 cm long. How far is $\overline{CD}$ from the center if the length of the radius is

1. 13 cm
2. 7 cm
3. 14 cm
4. 8 cm
5. 12 cm
6. 10 cm
7. $5\sqrt{2}$ cm
8. $3\sqrt{6}$ cm



C. Given: $\overline{CD}$ is 20 cm long. How long is the radius of the circle if the distance of $\overline{CD}$ from the center is

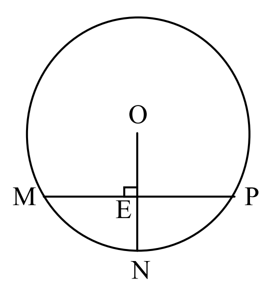
1. 7 cm
2. 10 cm
3. 13 cm
4. 8 cm
5. 5 cm
6. $\sqrt{21}$ cm
7. $5\sqrt{5}$ cm
8. $4\sqrt{6}$ cm

D. $\overline{AC}$ is 12 cm long. How long is chord $\overline{CD}$ if its distance from the center is

1. 10 cm
2. 6 cm
3. 8 cm
4. 5 cm
5. 9 cm
6. $\sqrt{23}$ cm
7. $2\sqrt{11}$ cm
8. $4\sqrt{5}$ cm

E. Solve the following problems.

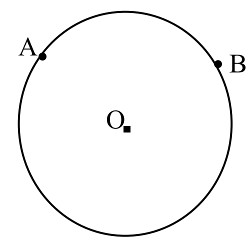
1. $\overline{ON} \perp \overline{MP}$
   $\overline{ME}$ = 7x + 5
   $\overline{PE}$ = 4x – 20
   Solve for $\overline{ME}$, $\overline{PE}$ and $\overline{MP}$



2. In a circle are two chords whose lengths are 10 cm and 24 cm respectively. If the radius of the circle is 13 cm, what is the maximum distance of the two chords? What is their minimum distance?

Lesson 2

Arcs and Central Angles

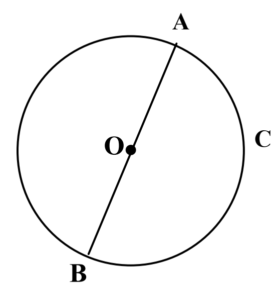


A part of a circle between any two points is an arc. In the figure, the set of points from A to B is an arc. A circle is in itself an arc. Arc of a circle is measured in terms of degrees.

The whole arc making up the circle measures 360°.

Any arc of a circle can belong to any of these three groups.

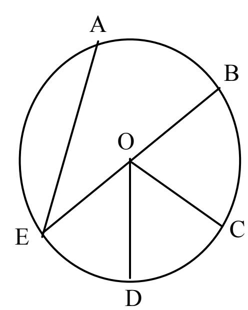
a. minor arc – an arc whose measure is between 0 and 180°.
b. semicircle – an arc whose measure is exactly 180°
c. major arc – an arc whose measure is between 180° and 360°



In the given figure, $\overline{AB}$ is a diameter, hence $\overset{\frown}{AB}$ represents a semicircle, $\overset{\frown}{AC}$ is minor arc and $\overset{\frown}{ABC}$ is a major arc. Aside from $\overset{\frown}{AC}$, another minor arc in the figure is $\overset{\frown}{BC}$. ACB also represents a semicircle.

Angles in a circle are formed by radii, chords, secants and tangents. Determination of the measures of the angles formed by these lines depends upon the measure of the intercepted arcs of the given angles.

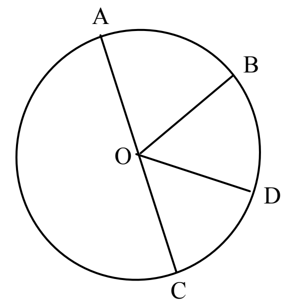
Examples:



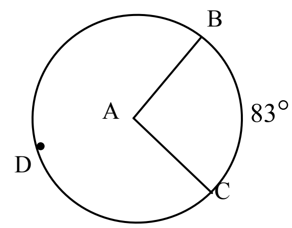
In circle some angles formed by chords and radii are shown. Each of the angles intercepts an arc defined by the endpoints contained on the sides of the angle.

$\angle AEB$ intercepts $\overset{\frown}{AB}$.
$\angle$BOC intercepts $\overset{\frown}{BC}$
$\angle$COD intercepts $\overset{\frown}{CD}$
$\angle$EOD intercepts $\overset{\frown}{ED}$
$\angle AEB$ intercepts $\overset{\frown}{AB}$.
$\angle AEB$ intercepts $\overset{\frown}{AB}$.
$\angle AEB$ intercepts $\overset{\frown}{AB}$.

At this point we will discuss in detail the kinds of angles formed in a circle, their characteristics and how to get their measures from the measures of the intercepted arcs. We will start with the angle formed by two radii.



<u>Central angle</u> is an angle formed by two radii and the vertex is the center of the circle. In the figure, $\angle$AOB, $\angle$BOD and $\angle$DOC are all examples of central angles. Each of these angles has its own intercepted arc. $\angle$AOB intercepts $\overset{\frown}{AB}$, $\angle$BOD intercepts $\overset{\frown}{BD}$ and $\angle$DOC intercepts $\overset{\frown}{DC}$.

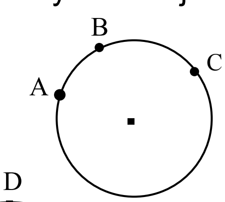


The measure of a central angle is numerically equal to its intercepted arc.

In the figure, $\angle$BAC is a central angle and $\angle$BAC intercepts $\overset{\frown}{BC}$. Since $m\overset{\frown}{BC}$ = 83, then $m\angle$BAC = 83, $m\overset{\frown}{BDC}$ = 277°.

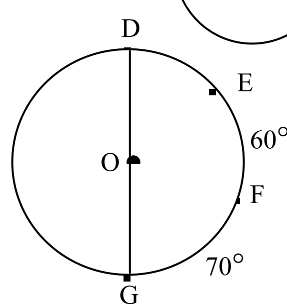
In the study of geometry, every new topic or concept is always associated with study of postulates, theorems and definitions. In the study of arcs and angles in a circle, we will discuss many theorems that will help us solve problems involving the said concepts. We will start with the simplest postulate in the chapter.

Like any measure, measure of an arc is also a unique real number and as such, we can perform the four fundamental operations on those measure. So the first postulate is the <u>Arc Addition Postulate:</u> The measure of an arc formed by two adjacent non-overlapping arcs is the sum of the measures of the two arcs.



In the given circle, $m\,\overset{\frown}{AC} = m\,\overset{\frown}{AB} + m\,\overset{\frown}{BC}$

Examples:



1. $\overline{DG}$ is a diameter. Find the measure of the following arcs.

   $\overset{\frown}{DG}$, $\overset{\frown}{DE}$, $\overset{\frown}{DF}$, $\overset{\frown}{GE}$, $\overset{\frown}{DGF}$

Solution:

Since $\overline{DG}$ is a diameter, then $\overset{\frown}{DG}$ is a semicircle.

Therefore,
$m\,\overset{\frown}{DG}$ = 180
$m\,\overset{\frown}{DE}$ = 180 – (60 + 70)
= 180 – 130
= 50

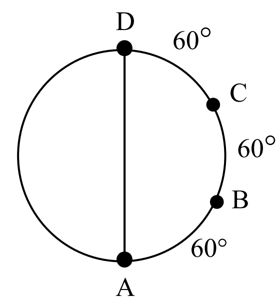
$m\,\overset{\frown}{DF} = m\,\overset{\frown}{DE} + m\,\overset{\frown}{EF}$
$= 50 + 60$
$= 110$

$m\,\overset{\frown}{GE} = m\,\overset{\frown}{GF} + m\,\overset{\frown}{FE}$
$= 70 + 60$
$= 130$

$m\,\overset{\frown}{DGF} = m\,\overset{\frown}{DG} + m\,\overset{\frown}{GF}$
$= 180 + 70$
$= 250$

Definitions:

In the same circle or in congruent circles, arcs which have the same measure are congruent.

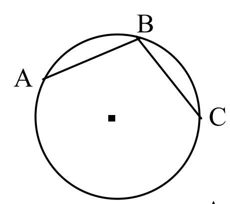


Example: 1. In the figure, $m\,\overset{\frown}{DC} = 60$, $m\,\overset{\frown}{BC} = 60$
$m\,\overset{\frown}{AB} = 60$.
Therefore, $\overset{\frown}{DC} \cong \overset{\frown}{BC} \cong \overset{\frown}{AB}$

2. Since every semicircle measures 180°, then all semicircles are congruent.

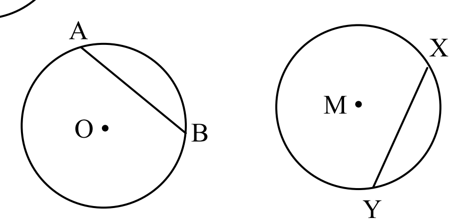
Theorem:

If two minor arcs of a circle or of congruent circles are congruent, then the corresponding chords are congruent.



Examples: 1. Given: $\overset{\frown}{AB} \cong \overset{\frown}{BC}$
Since $\overset{\frown}{AB}$ subtends $\overline{AB}$ and
$\overset{\frown}{BC}$ subtends $\overline{BC}$ then
$\overline{AB} \cong \overline{BC}$

2. Circle O $\cong$ circle M
If $\overset{\frown}{AB} \cong \overset{\frown}{XY}$, then $\overline{AB} \cong \overline{XY}$

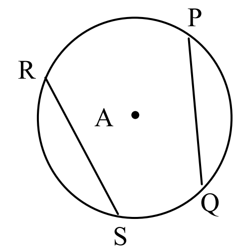


Theorem:

If two chords of a circle or of congruent circles are congruent, then the corresponding minor arcs are congruent.

This is the converse of the previous theorem. Basically if you prove these two theorems, the steps will be just the reverse of the other. Instead of proving them, showing examples will be more beneficial to you.

In circle A, if $\overline{RS} \cong \overline{PQ}$
then $\widehat{RS} \cong \widehat{PQ}$

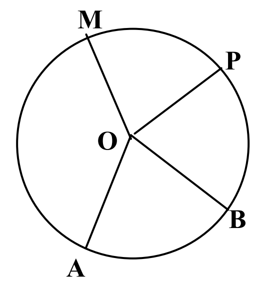


Theorem:

If two central angles of a circle or of congruent circles are congruent, then the corresponding minor arcs are congruent.

Example: In circle O, $\angle MNO \cong \angle BOA$

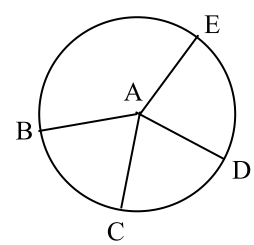
Therefore, $\widehat{MP} \cong \widehat{AB}$



Theorem:

If two minor arcs of a circle or of congruent circles are congruent, then the corresponding central angles are congruent.

Example:

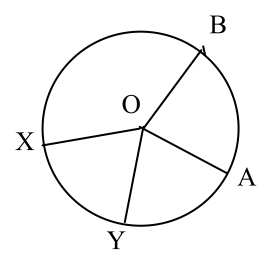


In circle A, $BC \cong DE$
Therefore $\angle BAC \cong \angle DAE$`

Theorem:

If two central angles of a circle or of congruent circles are congruent, then the corresponding chords are congruent.

Given: In circle O, $\angle XOY \cong \angle AOB$
Prove: $\overline{XY} \cong \overline{AB}$



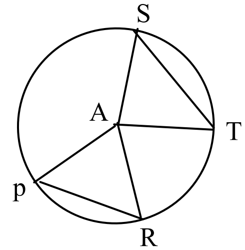
Proof

Statements	Reasons
1. In circle O, $\angle XOY \cong \angle AOB$	1. Given
2. $\overline{OX} \cong \overline{OB}$, $\overline{OY} \cong \overline{OA}$	2. Radii of the same or congruent circles are congruent
3. $\Delta XOY \cong \Delta BOA$	3. SAS congruency Postulate
4. $\overline{XY} \cong \overline{AB}$	4. Corresponding parts of congruent triangles are congruent..

Theorem:

If two chords of a circle or of congruent circles are congruent circles are congruent, then the corresponding central angles are congruent.

Given: In circle A, $\overline{PR} \cong \overline{ST}$
Prove: $\angle PAR \cong \angle SAT$

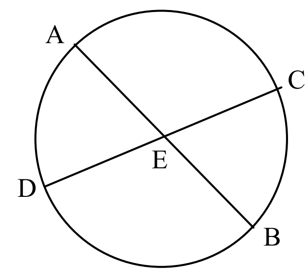


Proof:

Statements	Reasons
1. In circle A, $\overline{PR} \cong \overline{ST}$	1. Given
2. $\overline{AP} \cong \overline{AS}$ $\overline{AR} \cong \overline{AT}$	2. Radii of the same circle are congruent.
3. $\Delta PAR \cong \Delta SAT$	3. SSS Congruency Postulate
4. $\angle PAR \cong \angle SAT$	4. Corresponding parts of congruent triangles are congruent

Examples:

Given: $\overline{AB}$ *and* $\overline{CD}$ are diameters of circle E.
1. What is true about $\angle AED$ and $\angle BEC$? Why?
2. What kind of angles are they?
3. Give as many conclusions as you can based on the previously discussed theorems.



Answers:

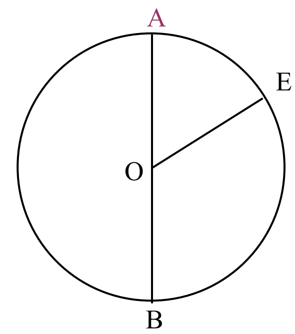
1. $\angle AED \cong \angle BEC$. They are vertical angles and vertical angles are congruent.
2. In the circle they are central angles. Central angles are angles whose vertex is the center of the circle.
3. a. $\overset{\frown}{AD} \cong \overset{\frown}{BC}$. If two central angles of a circle or of congruent circles are congruent, then the corresponding arcs are congruent.
   b. $\overline{AD} \cong \overline{BC}$

Likewise
1. $\angle AEC \cong \angle BED$
2. $\overset{\frown}{AC} \cong \overset{\frown}{DB}$
3. $\overline{AC} \cong \overline{DB}$

Try this out

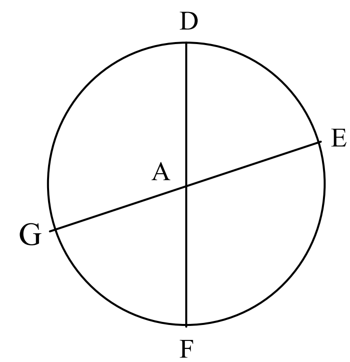
A. $\overline{AB}$ is a diameter of circle O.
$m\angle AOE = 82$.

Find the measures of:



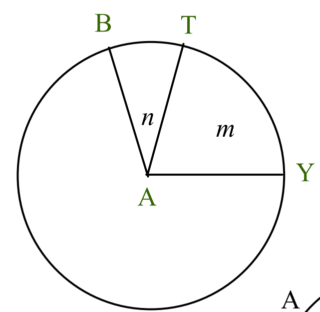
1. $\overset{\frown}{AB}$
2. $\overset{\frown}{AE}$
3. $\overset{\frown}{BE}$
4. $\overset{\frown}{ABE}$
5. $\overset{\frown}{BAE}$

B. $\overline{GE}$ and $\overline{FD}$ are diameters of circle A. If $\overset{\frown}{DA}$ = 73º, find the measures of

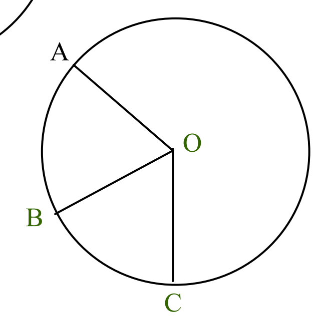


1. $\angle$DAE
2. $\angle$GAF
3. $\angle$EAF
4. $\angle$DAG
5. $\overset{\frown}{GF}$
6. $\overset{\frown}{DG}$
7. $\overset{\frown}{FDE}$

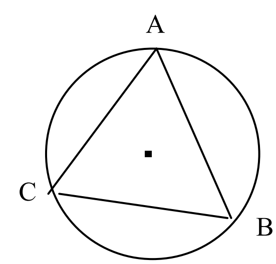
C. Given circle A. If $m\,\overset{\frown}{BTY}$ = 116,
and $m = 3n$, find



1. m
2. n
3. $\overset{\frown}{BT}$
4. $\overset{\frown}{TY}$
5. $m\angle$BAT
6. $m\angle$TAY
7. $m\angle$BAY



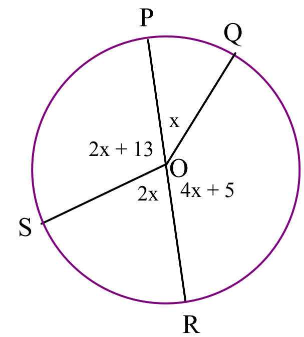
D. Given circle O. $\overset{\frown}{AB} \cong \overset{\frown}{BC}$. If $m\overset{\frown}{AB}$ = 56, what is $m\angle AOB$? What is $m\overset{\frown}{ABC}$?
Which chords are congruent?



E. A. B and C are three points on the circle.
IF $\overset{\frown}{AC} \cong \overset{\frown}{AB} \cong \overset{\frown}{BC}$, what is the measure of each arc?
What is true about the chords $\overline{AC}, \overline{AB}$, and $\overline{BC}$?
If ABC is 16 more than three times $\overset{\frown}{AC}$,
find $m\overset{\frown}{AC}$, $m\overset{\frown}{ABC}$.

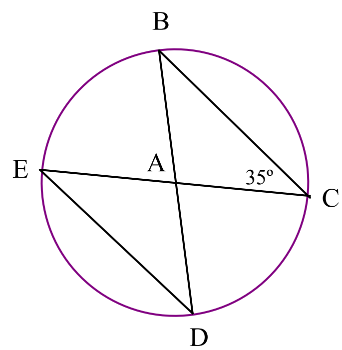
F. P, Q and R are three points on a circle. If the ratio $\overset{\frown}{PQ}$:$\overset{\frown}{QR}$:$\overset{\frown}{PR}$ = 3:4:5, find the measures of $\overset{\frown}{PR}$, $\overset{\frown}{QR}$ and $\overset{\frown}{PS}$.

G. Using the figure and the given in it, find the measures of:



1. $\overset{\frown}{PQ}$
2. $\overset{\frown}{QR}$
3. $\overset{\frown}{SR}$
4. $\overset{\frown}{PS}$
5. $\angle POQ$
6. $\angle QOR$
7. $\angle SOR$
8. $\angle POS$

H. $\overline{BD}$ and $\overline{EC}$ are diameters of circle A. If $m\angle C = 35$, find the measures of



1. $\angle B$
2. $\angle E$
3. $\angle D$
4. $\angle BAC$
5. $\angle EAD$
6. $\overset{\frown}{BC}$
7. $\overset{\frown}{CD}$

Lesson 3

Arcs and Inscribed Angles

Another angle in a circle that is very important in the study of circle is the inscribed angle.

Definition:

An <u>inscribed angle</u> is an angle whose vertex lies on the circle and the sides contain chords of the circle.

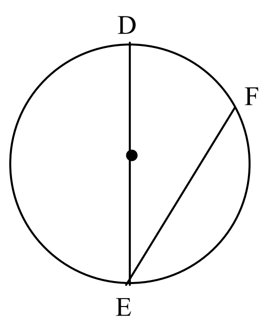


Fig. 1

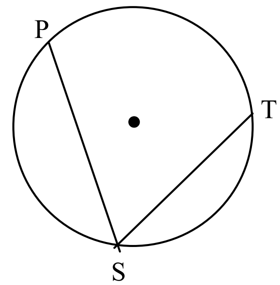


Fig. 2

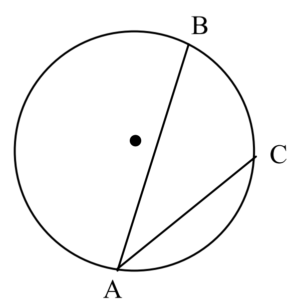


Fig 3

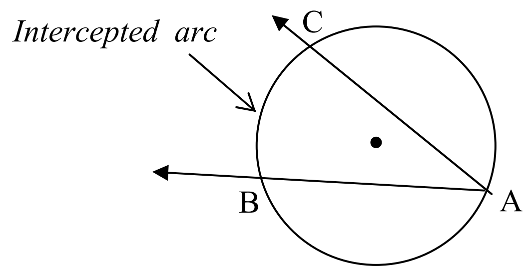
Each of the angle shown above is an example of an inscribed angle. Three cases are represented here relative to the position of the sides in relation to the center of the circle.

Case 1. the center of the circle is on one side of the inscribed angle.
Case 2, the center of the circle is in the interior of the inscribed angle.
Case 3, the center of the circle is on the exterior of the inscribed angle.

In the study of the angles in a circle and in determining their measures, it is important to determine the intercepted arc(s) of the given angle. To understand better, let us see some examples.

In the figure, the arc in the interior of the angle is the <u>intercepted arc</u> of the angle.

The intercepted arc of $\angle BAC$ is the minor arc $\overset{\frown}{AC}$.



In the given examples of inscribed angles above the following holds:

a) In figure 1, $\angle DEF$ is an inscribed angle
$\angle DEF$ intercepts arc $\overset{\frown}{DF}$

b) In figure 2, $\angle PST$ is an inscribed angle,
$\angle PST$ intercepts arc $\overset{\frown}{PT}$

c) In figure 3, $\angle BAC$ is an inscribed angle
$\angle BAC$ intercepts arc $\overset{\frown}{BC}$

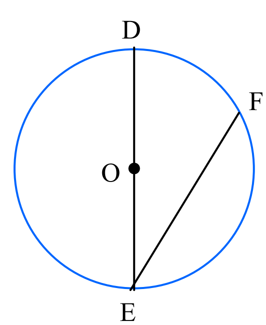
Every angle whether in a circle on in any plane is associated with a unique number defined as its measure. If the measure of a central angle is equal to the measure of its intercepted arc, the next theorem will tell us how to find the measure of the inscribed angle.

Theorem: Inscribed angle Theorem

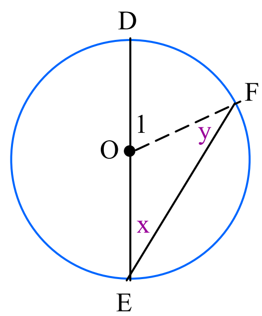
The measure of an inscribed angle is equal to one half the measure of its intercepted arc.

It means that in the given figure,

$$m\angle DEF = \tfrac{1}{2}\, m\overset{\frown}{DF}$$



Since there are three cases by which an inscribed angle can be drawn in a circle, then we have to prove each of those cases.



Case 1 (One side of the angle is the diameter of the circle)

Given: Circle O with inscribed angle $\angle DEF$
Use the notation in the figure for clarity

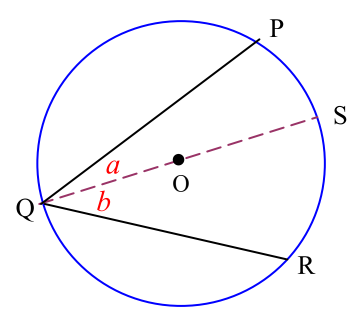
Prove: $m\angle DEF = \frac{1}{2}(m\widehat{DF})$

Proof:

Statements	Reasons
1. Circle O with inscribed angle $\angle DEF$	1. Given
2. Draw $\overline{OF}$ to form ΔFOE	2. Line determination postulate
3. $\angle$ 1 is an exterior angle of ΔFOE	3. Definition of exterior angle
4. $m\angle$ 1 = x + y	4. Exterior angle theorem
5. $\overline{OF} \cong \overline{OE}$	5. Radii of the same circle are congruent
6. ΔFOE is an isosceles triangle	6. Definition of isosceles triangle
7. x = y	7. Base angles of isosceles triangle are congruent
8. $m\angle$ 1 = x + x = 2x	8. Substitution (Steps 4 and 7)
9. 2x = $m\angle$ 1, x = ½ $m\angle$ 1	9. Multiplication property of equality
10. But $\angle$ 1 is a central angle	10. Definition of central angle
11. $m\angle$ 1 = $m\,\widehat{DF}$	11. Measure of a central angle equals its intercepted arc.
12. x = $m\angle DEF = \frac{1}{2}(m\widehat{DF})$	12. Substitution (Steps 9 and 11)

So, we have proven case 1. Let us now prove case 2 of the inscribed angle theorem.

Case 2. (The center of the circles lies in the interior of the inscribed angle)



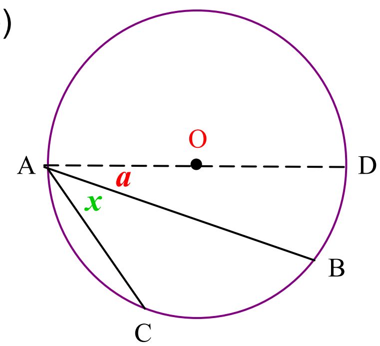
Given : Circle O with inscribed $\angle PQR$

Prove: $m\angle PQR = \frac{1}{2}\ m\,\widehat{PR}$

Proof:

Statements	Reasons
1. Circle O with inscribed $\angle PQR$. Use the given notation in the figure.	1. Given
2. Draw diameter $\overline{QS}$	2. Line determination Postulate
3. $m\angle PQR$ = a + b	3. Angle Addition Postulate
4. a = $\frac{1}{2} m\overset{\frown}{PS}$ b = $\frac{1}{2} m\overset{\frown}{SR}$	4. Inscribed angle theorem (Case 1)
5. a + b = $\frac{1}{2} m\overset{\frown}{PS}$ + $\frac{1}{2} m\overset{\frown}{SR}$ = $\frac{1}{2}(m\overset{\frown}{PS} + m\overset{\frown}{SR})$	5. Addition Property of Equality
6. $m\overset{\frown}{PR}$ = $m\overset{\frown}{PS}$ + $m\overset{\frown}{PR}$	6. Arc Addition Postulate
7. $m\angle PQR$` = $\frac{1}{2}(m\overset{\frown}{PS} + m\overset{\frown}{PR})$	7. Transitive Property of Equality
8. $m\angle PQR$` = $\frac{1}{2}$ $m\overset{\frown}{PR}$	8. Transitive Property of Equality

Case 3. (The center is in the exterior of the inscribed angle)



Given: $\angle BAC$ is an inscribed angle in circle O
Use the additional notation in the figure

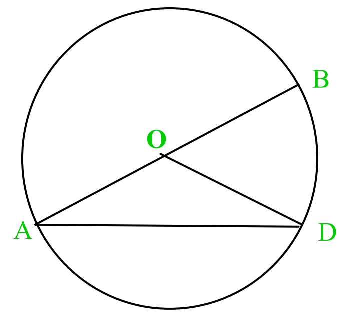
Prove: $m\angle BAC = \frac{1}{2} m\overset{\frown}{BC}$

Proof:

Statements	Reasons
1. Draw diameter $\overline{AD}$	1. Line determination Postulate
2. $m\angle DAC = m\angle DAB + m\angle BAC$	2. Angle Addition Postulate
3. $m\angle BAC = m\angle DAC - m\angle DAB$	3. Subtraction Property of Equality
4. $m\angle DAC = \frac{1}{2} m\overset{\frown}{DC}$ $m\angle DAB = \frac{1}{2} m\overset{\frown}{DB}$	4. Inscribed angle Theorem (Case 1)
5. $m\angle BAC = \frac{1}{2} m\overset{\frown}{DC} - \frac{1}{2} m\overset{\frown}{DB} = \frac{1}{2}(m\overset{\frown}{DC} - m\overset{\frown}{DB})$	5. Substitution
6. $m\overset{\frown}{DC}$ = $m\overset{\frown}{DB}$ + $m\overset{\frown}{BC}$	6. Arc Addition Postulate
7. $m\overset{\frown}{BC}$ = $m\overset{\frown}{DC}$ – $m\overset{\frown}{DB}$	7. Subtraction Property of Equality
8. $m\angle BAC = \frac{1}{2} m\overset{\frown}{BC}$	8. Substitution

From the proofs that were given, we can therefore conclude that wherever in the circle the inscribed angle is located, it is always true that its measure is one-half its intercepted arc.

Examples. Use the figure at the right.



1. Given: circle O. $m\angle BOD = 80$

   Find: $m\widehat{BD}$, $m\angle BAD$

   Solution:

   Since $m\angle BOD = 80$, then

   a. $m\widehat{BD} = 80$

   b. $m\angle BAD = \frac{1}{2}\widehat{BD}$

   $= \frac{1}{2}(80)$

   $= 40$

2. Given: circle O. $m\angle BAD = 37$

   Find: $m\widehat{BD}$, $m\angle BOD$

   Solution:

   $m\angle BAD = 37 = \frac{1}{2}m\widehat{BD}$

   $m\widehat{BD} = 2(37) = 74$

   $m\angle BOD = m\widehat{BD}$

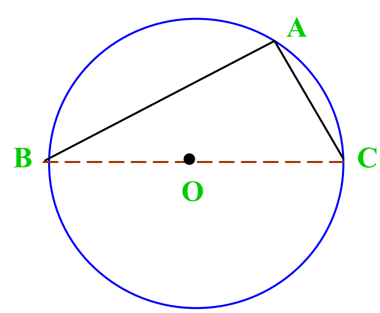
   $m\angle BOD = 74$

Like in the study of central angles and its measure, discussing inscribed angles and its measure also involves many theorems. Each previous theorem studied is always a tool in proving the next theorem.

The following theorem is one of the most useful theorem in solving problems which involve inscribed angles.

Theorem: Angle in a semicircle theorem.

An angle inscribed in a semicircle is a right angle.



Given: Circle O. BAC is a semicircle.

Prove: $\angle BAC$ is a right angle. ($m\angle BAC = 90$

Proof:

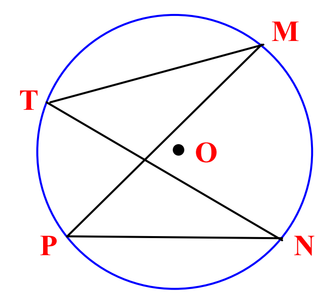
Statements	Reasons
1. Draw $\overline{BC}$ passing through center O. 2. $\angle ABC$, $\angle ACB$, and $\angle BAC$ are all inscribed angles. 3. $m\angle ABC = \frac{1}{2}\widehat{AC}$, $m\angle ACB = \frac{1}{2}\widehat{AB}$ 4. $m\widehat{BAC} = m\widehat{AC} + m\widehat{AB}$ 5. $\widehat{BAC}$ is a semicircle 6. $m\widehat{BAC} = 180$ 7. $m\widehat{AC} + m\widehat{AB} = 180$ 8. $m\angle ABC + m\angle ACB = \frac{1}{2}\widehat{AC} + \frac{1}{2}\widehat{AB} = \frac{1}{2}(\widehat{AC} + \widehat{AB})$ 9. $m\angle ABC + m\angle ACB = \frac{1}{2}(180) = 90$ 10. $m\angle ABC + m\angle ACB + m\angle BAC = 180$ $m\angle ABC + m\angle ACB = 90$ 11. $m\angle BAC = 90$ 12. $\angle BAC$ is right angle	1. Definition of diameter 2. Definition of inscribed angles 3. Inscribed Angle Theorem 4. Arc Addition Postulate 5. Given 6. The measure of a semicircle is 180 7. Transitive Property of Equality 8. Addition Property of Equality (Step 3) 9. Substitution (Steps 7 and 8) 10. The sum of the angles of a triangle is 180. 11. Subtraction Property of Equality (Step 10 – step 9) 12. Definition of a right angle

From this point onward, you can use this very important theorem in proving or in exercises.

There are other theorems on inscribed angle that are also important as the previous theorem. Of those theorems, we will prove two and the rest, you can answer as exercises.

Theorem:

Inscribed angles subtended by the same arc are congruent.



Given: Circle O. $\widehat{MN}$ subtends both $\angle$T and $\angle$P
$\angle$T and $\angle$P are inscribed angles

Prove: $\angle T \cong \angle P$

Proof:

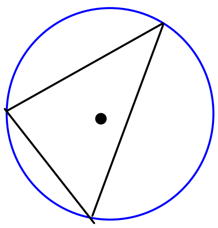
Statements	Reasons
1. In circle O, $\widehat{MN}$ subtends both $\angle$T and $\angle$P. $\angle$T and $\angle$P are inscribed angles. 2. $m\angle T = \frac{1}{2} m\widehat{MN}$ $m\angle P = \frac{1}{2} m\widehat{MN}$ 3. $m\angle T = m\angle P$ 4. $\angle T \cong \angle P$	1. Given 2. Inscribed Angle Theorem 3. Transitive Property of Equality 4. Definition of congruent angles

The next theorem is about polygon inscribed in a circle.

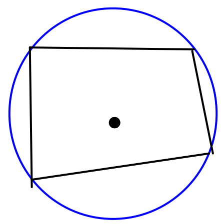
Definition:

A polygon inscribed in a circle is polygon whose vertices lie on the circle.

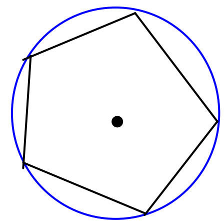
Examples: The figures below show examples of <u>inscribed polygon.</u>



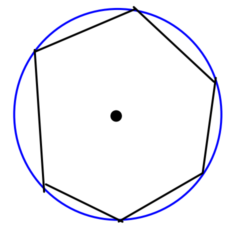
Inscribed triangle



Inscribed Quadrilateral

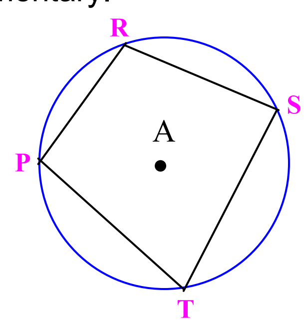


Inscribed Pentagon



Inscribed Hexagon

Theorem: Opposite angles of a inscribed quadrilateral are supplementary.



Given: Circle A. PRST is an inscribed quadrilateral.

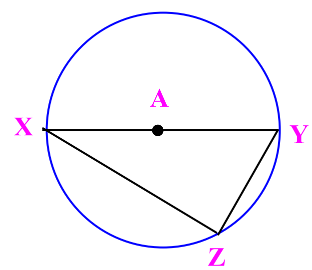
Prove: $\angle$P and $\angle$S are supplementary
$\angle$R and $\angle$T are supplementary

Proof:

Statements	Reasons
1. Circle A. PRST is an inscribed quadrilateral.	1. Given
2. $m\angle P = \frac{1}{2}\, m\widehat{RST}$ $m\angle S = \frac{1}{2}\, m\widehat{RPT}$ $m\angle R = \frac{1}{2}\, m\widehat{PTS}$ $m\angle T = \frac{1}{2}\, m\widehat{PRS}$	2. Inscribed angle theorem
3. $m\angle P + m\angle S = \frac{1}{2}\, m\widehat{RST} + \frac{1}{2}\, m\widehat{RPT}$	3. Addition property of equality
4. $m\angle P + m\angle S = \frac{1}{2}(m\widehat{RST} + m\widehat{RPT})$	4. Factoring
5. $m\widehat{RST} + m\widehat{RPT} = 360$	5. The arc of the whole circle is 360º
6. $m\angle P + m\angle S = \frac{1}{2}(360)$	6. Substitution (Steps 4 and 5)
7. $m\angle P + m\angle S = 180$	7. Algebraic process (step 6)
8. $\angle$P and $\angle$S are supplementary	8. Definition of supplementary angles
9. $m\angle R + m\angle S = \frac{1}{2}\, m\widehat{PTS} + \frac{1}{2}\, m\widehat{PRS}$	9. Addition property of equality
10. $m\angle R + m\angle S = \frac{1}{2}\,(m\widehat{PTS} + m\widehat{PRS})$	10. Factoring
11. $m\widehat{PTS} + m\widehat{PRS} = 360$	11. The arc of the whole circle is 360º
12. $m\angle R + m\angle S = \frac{1}{2}(360)$	12. Substitution (Steps 4 and 5)
13. $m\angle R + m\angle S = 180$	13. Algebraic process (step 6)
14. $\angle$R and $\angle$T are supplementary	14. Definition of supplementary angles

Examples:

1. Given: $\overline{XY}$ is a diameter.
   a. What kind of angle is $\angle Z$?
   b. If $m\angle X = 35$, what is $m\angle Y$?
   c. If $m\angle Y = 73$, what is $m\overset{\frown}{XZ}$? What is $m\overset{\frown}{YZ}$?



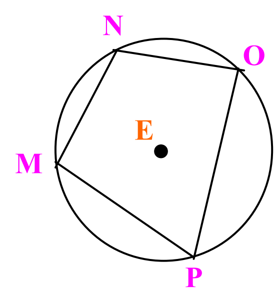
Answers:

a. Since $\overline{XY}$ is a diameter, then $\overset{\frown}{XZY}$ is a semicircle and $\angle Z$ is inscribed in a semicircle. Therefore, $\angle Z$ is a right angle.

b. $m\angle X + m\angle Y = 90.$
$m\angle Y = 90 - m\angle X$
$m\angle Y = 90 – 35$
$m\angle Y = 65$

c. $\angle Y$ intercepts $\overset{\frown}{XZ}$.
$m\overset{\frown}{XZ} = 2(75) = 150$
$m\overset{\frown}{YZ} = 180 – 150$
$m\overset{\frown}{YZ} = 30$

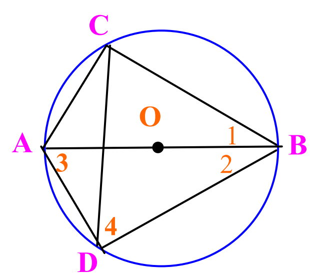
2. MNOP is inscribed in circle E. If $m\angle M = 94$, what is $m\angle O$?



Answer:

$\angle M$ and $\angle O$ are supplementary.
$m\angle M + m\angle O = 180$
$m\angle O = 180 - m\angle M$
$= 180 – 94$
$= 86$

3. Given: Circle O. $\overline{AB}$ is a diameter
$m\angle 1 = 36$ and $m\angle 3 = 61$.
Find: $m\angle 2$, $m\angle 4$, $m\angle CBD$,
$m\angle ADB$, $m\angle ACB$, $m\overset{\frown}{CBD}$,
$m\angle CAD$, $m\overset{\frown}{AD}$



Solution:

$m\angle 1 = 36$, $m\overset{\frown}{AC} = 2(36) = 72$
$m\angle 3 = 61$, $m\overset{\frown}{BD} = 2(61) = 122$
$m\angle 2 = ½\ \overset{\frown}{AD}$
$m\overset{\frown}{AD} = 180 – \overset{\frown}{BD}$
$= 180 – 122 = 58$
$m\angle 2 = ½\ (58)$
$= 29$

$m\angle 4 = ½\,\overset{\frown}{CB}$
$m\overset{\frown}{CB} = 180 - m\overset{\frown}{AC}$
$= 180 - 72$
$= 108$
$m\angle 4 = ½\,(108)$
$= 54$

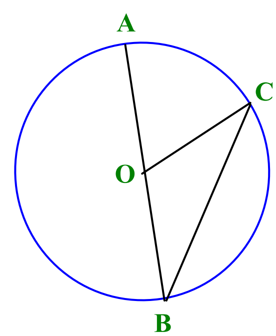
$m\angle CBD = ½(m\,\overset{\frown}{AC} + m\overset{\frown}{AD})$
$= ½(72 + 58)$
$= ½(130)$
$= 65$

$m\angle ADB = 90$ (Angle in a semicircle)
$m\angle ACB = 90$ (angle in a semicircle)

$m\overset{\frown}{CBD} = m\overset{\frown}{CB} + m\overset{\frown}{BD}$
$= 108 + 122$
$= 230$

$m\angle CAD = ½(m\overset{\frown}{CBD})$
$= ½\,(230)$
$= 115$

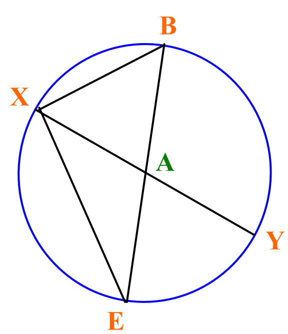
Try this out



A. Given: $\overline{AB}$ is a diameter of circle O.
$m\overset{\frown}{AC} = 79$.
Find:

1. $m\angle AOC$
2. $m\angle ABC$
3. $m\angle COB$

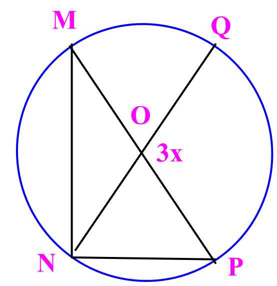
B. Given: Circle A., $\overline{XY}$ and $\overline{BE}$ are diameters
$m\angle XAE = 104$.
Find:



4. $m\,\overset{\frown}{XE}$
5. $m\,\overset{\frown}{BX}$
6. $m\angle E$
7. $m\angle B$
8. $m\angle BXY$
9. $m\angle YXE$

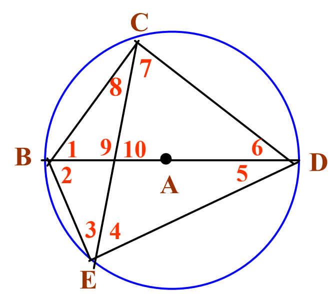
C. Using the given figure, find:

10. x
11. $m\angle$ MNQ
12. $m\angle$ MOQ
13. $m\angle$ POQ
14. $m\angle$ M
15. $m\angle$ MON



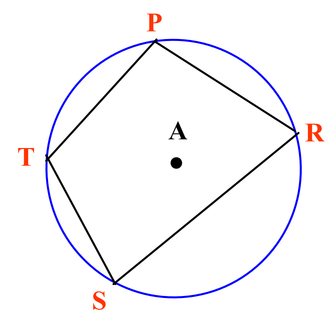
D. $\overline{BD}$ is a diameter of circle A.
If $m\overset{\frown}{BC} = 78$, and $m\overset{\frown}{DE} = 132$,
find:

16. $m\overset{\frown}{CD}$
17. $m\overset{\frown}{BE}$
18. $m\angle 1$
19. $m\angle 2$
20. $m\angle 3$
21. $m\angle 4$
22. $m\angle 5$
23 $m\angle 6$
24. $m\angle 7$
25. $m\angle 8$
26. $m\angle 9$
27. $m\angle 10$

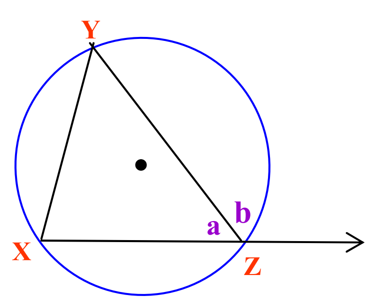


E. PRST is inscribed in circle A.
If $m\angle T = (5x - 4)^{\circ}$ and $m\angle R = (4x + 13)^{\circ}$
find:

28. x
29. $m\angle$ T
30. $m\angle$ R



F. ΔXYZ is inscribed in the circle.
If $\overline{XY} \cong \overline{XZ}$, prove that $m\angle X = b - a$



Let's Summarize

1. A circle is the set of all points that are at the same distance from a given point in the plane.

2. Some of the lines associated with circle are the following:

    a. Radius
    b. Chord
    c. Diameter
    d. Secant
    e. Tangent

3. If a radius is perpendicular to a chord, then it bisects the chord.

4. If a radius of a circle bisects a chord that is not a diameter, then it is perpendicular to the chord.

5. Congruent circles are circles that have congruent radii. Concentric circles are circles having the same center.

6. A circle is made up of arcs classified as minor arc, semicircle and major arc.

7. A central angle is an angle on the circle whose vertex is the center of the circle.

8. The measure of the central is numerically equal to its intercepted arc.

9. If two minor arcs of a circle or of congruent circle are congruent, then,

    a. the corresponding central angles are congruent,
    b. the corresponding subtended chords are congruent

10. An inscribed angle is an angle on the circle whose vertex is a point on the circle.

11. The measure of an inscribed angle is equal to one-half its intercepted arc.

12. An angle inscribed in a semicircle is a right angle.

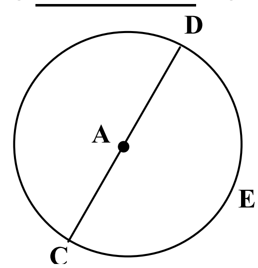
13. Inscribed angle subtended by the same arc are congruent.

14. The opposite angles of an inscribed quadrilateral are supplementary.

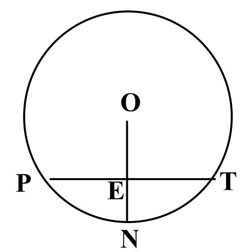
What have you learned

Answer as indicated.

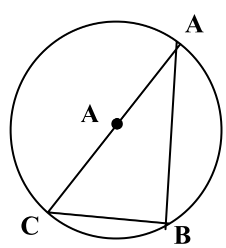
1. If the diameter of a circle is 15 cm, what is the length of the radius?
2. A line that intersects a circle at one point is called _________.
3. If a radius bisects a chord which is not a diameter, then its is _________ to the chord.



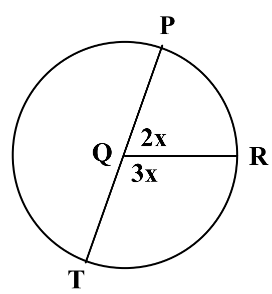
4. $\overline{CD}$ is a diameter of circle A. $\overset{\frown}{CED}$ is a ______________.
5. $\overset{\frown}{CE}$ is a _____________.
6. $\overset{\frown}{CDE}$ is a ______________



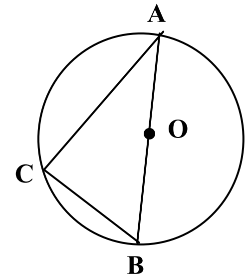
7. $\overline{ON} \perp \overline{PT}$ at E. If $\overline{OE}$ = 6 cm, and the radius of the circle is 10 cm, what is the length of $\overline{PT}$?.



8. $\overline{AC}$ is a diameter of circle O. Using the given in the given figure, find
   a. $m\angle$ A
   b. $m\angle$ C
   c. $m\ \overset{\frown}{AB}$
   d. $m\ \overset{\frown}{BC}$



9. $\overline{PT}$ is a diameter of circle Q. Find
   a. $m\angle$ PQR
   b. $m\angle$ RQT



10. ΔABC is inscribed in circle O. If the ratio of $m\angle$ A:$m\angle$ B:$m\angle$ C = 2:3:5, Find
    a. $m\ \overset{\frown}{BC}$
    b. $m\ \overset{\frown}{AC}$
    c. $m\ \overset{\frown}{AB}$

Answer Key

How much do you know

1. a. circle O
   b. $\overline{MN}$
   c. $\overline{MO}$, $\overline{ON}$
   d. $\overline{MT}$, $\overline{MR}$
   e. $\overleftrightarrow{MR}$
   f. $\overleftrightarrow{MS}$
2. bisects
3. minor arc
4. major arc
5. 4 cm
6. 5 cm
7. 93°
8. 46.5°
9. 98°, 49°
10. 77°

Try this out
Lesson 1

A. 1. circle O
   2. $\overline{AC}$, $\overline{BD}$
   3. $\overline{AD}$, $\overline{BC}$
   4. $\overleftrightarrow{EC}$, $\overleftrightarrow{BC}$
   5. $\overleftrightarrow{CF}$

B. 1. 12 cm
   2. $2\sqrt{6}$ cm
   3. $\sqrt{171}$ cm
   4. $\sqrt{43}$ cm
   5. $\sqrt{119}$ cm
   6. $5\sqrt{3}$ cm
   7. 5 cm
   8. $\sqrt{29}$ cm

C. 1. $\sqrt{149}$ cm
   2. $10\sqrt{2}$ cm
   3. $\sqrt{269}$ cm
   4. $2\sqrt{41}$ cm
   5. $5\sqrt{5}$ cm
   6. 11 cm
   7. 15 cm
   8. 14 cm

D. 1. $4\sqrt{11}$ cm
2. $12\sqrt{3}$ cm
3. $8\sqrt{5}$ cm
4. $2\sqrt{19}$ cm
5. $6\sqrt{7}$ cm
6. 22 cm
7. 20 cm
8. 16

Problem Solving:
1. $\overline{ME}$ = 52, $\overline{PE}$ = 52, $\overline{MP}$ = 104
2. maximum distance is 12 cm
   minimum distance is 2 cm

Lesson 2

A. 1. 180º
2. 82º
3. 98º
4. 278º
5. 262º

B. 1. 73º
2. 73º
3. 107º
4. 107º
5. 73º
6. 107º
7. 253º

C. 1. 29º
2. 87º
3. 29º
4. 87º
5. 29º
6. 87º
7. 116º

D. 1. 56
2. 56
3. $\overline{AB}$ and $\overline{CD}$

E. 1. Each arc measures 120
2. $\overset{\frown}{AC}$ = 86º
3. $\overset{\frown}{ABC}$ = 274º

F. $\overset{\frown}{PQ}$ = 90
$\overset{\frown}{QR}$ = 120
$\overset{\frown}{PR}$ = 150

G. 1. 38º
2. 157º
3. 76º
4. 89º
5. 38º
6. 157º
7. 76º
8. 89º

H. 1. 35º
2. 35º
3. 35º
4. 110º
5. 110º
6. 110º
7. 17º

Lesson 3

A. 1. 79
2. 39.5
3. 101

B. 4. 104
5. 76
6. 38
7. 52
8. 52
9. 38

C. 10. 36
11. 36
12. 72
13. 108
14. 36
15. 108

D. 16. 102
17. 48
18. 51
19. 66
20. 39
21. 51
22. 24
23. 39
24. 66
25. 24
26. 105
27. 75

E. 28. 19
29. 91
30. 89

F. Proof:

1. $\angle X + \angle Y + a = 180$	1. Sum of the measures of the angles of a triangle is 180.
2. $b = m\angle X + m\angle Y$	2. Exterior angle theorem
3. $m\angle Y = a$	3. Angles opposite equal sides in the same triangle are congruent
4. $b = m\angle X + a$	4. Substitution
5. $m\angle X = b - a$	5. Subtraction Property of Equality

What have you learned

1. 7.5 cm
2. tangent
3. perpendicular
4. semicircle
5. minor arc
6. major arc
7. 16 cm
8. a. 22.5
   b. 67.5
   c. 135
   d. 45
9. a. 72
   b. 108
10. a. 72
    b. 108
    c. 180